SAMPLE OF THE STUDY MATERIAL
PART OF CHAPTER 1
Introduction to Linear Algebra

1.1. Introduction

Linear algebra is a specific branch of mathematics dealing with the study of vectors, vector spaces with functions that input one vector and output another vector and Eigen-value problems.

Such functions connecting input vector & output vector are called linear maps (or linear transformations or linear operators) and can be represented by matrices. The matrix theory is often considered as a part of linear algebra. Linear algebra is commonly restricted to the case of finite dimensional vector spaces, while the peculiarities of the infinite dimensional case are traditionally covered in linear functional analysis.

Linear algebra is heart of modern mathematics and its applications, such as to find the solution of a system of linear equations. Linear algebra has a concrete representation in analytic geometry and is generalized in operator theory and in module theory. It has extensive applications in engineering, physics, natural sciences, computer science, and the social sciences. It is extensively used in image processing, artificial intelligence, missile dynamics etc. Nonlinear mathematical models are often approximated by linear ones and solved through Linear algebra.

A very popular software, with which most of the engineers are familiar, called MATLAB. Matlab is an acronym for Matrix Laboratory. The whole basis of the software itself is matrix computation.

1.2. Definition of Matrix

A system of "*m n*" numbers arranged along m rows and n columns is called matrix of order m x n.

Traditionally, single capital letter is used to denote matrices and written in the form below:

$$A = [a_{ij}] = \begin{bmatrix} a_{11} & a_{12} & - & - & a_{1j} & - & - & a_{1n} \\ a_{21} & a_{22} & - & - & a_{2j} & - & - & a_{2n} \\ - & - & - & - & a_{ij} & - & - & a_{in} \\ a_{m1} & a_{m2} & - & - & - & - & - & a_{mn} \end{bmatrix}$$

Where element a_{ij} *indicates* i^{th} row & j^{th} column

1.3. Classification of Matrices

1.3.1. Row matrix

Def: A row Matrix is the matrix having single row (also it is called as row vector). For example
A = [12, 0, 80.9, 5] is row matrix

1.3.2. Column matrix

Def: A Column Matrices is the matrix having single column (also called column vector)

For example B= $\begin{bmatrix} 1 \\ 11.3 \\ 2 \\ 5 \end{bmatrix}$ is a Column Matrix

1.3.3. Square matrix

Def: A square matrix is a matrix having same number of rows and columns.

1.3.3.1. Order of square matrix:

Def: Order of Square matrix is no. of rows or columns

Let's see it through an example:

$$P = \begin{bmatrix} 7 & 9 & 8 \\ 1 & 3 & 6 \\ 9 & 7 & 5 \end{bmatrix} ;$$

Here, order of this matrix is 3

1.3.3.2. Principal Diagonal / Main diagonal / Leading diagonal of a Matrix:

Def: The diagonal of a square matrix (from the top left to the bottom right) is called as principal diagonal.

In the above example, diagonal connecting elements 7, 3 & 5 is a principal diagonal.

1.3.3.3. Trace of a Matrix:

Def: The sum of the diagonal elements of a square matrix is called the trace.
Trace is defined only for the square matrix.

Note: Some of the following results can be seen quite obviously.

- ✓ tr(A+B) = tr(A) + tr (B)
- ✓ tr(AB) = tr(BA)
- ✓ tr(β A) = β tr(A) , for a scalar β

1.3.4. Rectangular Matrix

Def: A rectangular matrix is a matrix having unequal number of rows and columns.

In other words, for a rectangular matrix, number of rows ≠ number of columns

1.3.5. Diagonal Matrix

Def: A Square matrix in which all the elements except those in leading diagonal are zero.

For example, $P=\begin{bmatrix}1 & 0 & 0\\ 0 & -2 & 0\\ 0 & 0 & 3\end{bmatrix}$

This matrix is sometimes written as P= diag[1, -2, 3]

Question: Is the following matrix a diagonal?

$$\begin{bmatrix}0 & 0 & 0\\ 0 & 2 & 0\\ 0 & 0 & 0\end{bmatrix}$$

Solution: Yes, (but why?)

Properties of diagonal matrix:

- ✓ $\text{diag}[a, b, c] + \text{diag}[x, y, z] = \text{diag}[a + x, b + y, c + z]$
- ✓ $\text{diag}[a, b, c] \times \text{diag}[x, y, z] = \text{diag}[ax, by, cz]$
- ✓ $\text{diag}[a, b, c]^{-1} = \text{diag}[a^{-1}, b^{-1}, c^{-1}]$
- ✓ $\text{diag}[a, b, c]^{n} = \text{diag}[a^{n}, b^{n}, c^{n}]$
- ✓ $\text{diag}[a, b, c]^{T} = \text{diag}[a, b, c]$ {Here T implies transpose}

1.3.6. Scalar Matrix

Def: A Diagonal matrix in which all the leading diagonal elements are same is called scalar matrix.

For example,

$$P=\begin{bmatrix}4 & 0 & 0\\ 0 & 4 & 0\\ 0 & 0 & 4\end{bmatrix}$$

1.3.7. Null Matrix (or Zero Matrix)

Def: A matrix is called Null Matrix if all the elements are zero.

For example, $O=\begin{bmatrix}0 & 0 & 0\\ 0 & 0 & 0\end{bmatrix}$ or $\begin{bmatrix}0 & 0 & 0\\ 0 & 0 & 0\\ 0 & 0 & 0\end{bmatrix}$

Note: No. of rows need not be equal to no. of columns

1.3.8. Identity Matrix / Unit Matrix

Def: A Diagonal matrix in which all the leading diagonal elements are '1' is called unit matrix.

For example, $I_3 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

1.3.9. Symmetric matrix

Def: A matrix is called Symmetric, if $a_{ij} = +a_{ji}$ for all i and j.

In other words, transpose of matrix is equal to the matrix itself

i.e. $\mathbf{A^T = A}$

It may be noted that the diagonal elements of the matrix can be anything.

1.3.10. Skew symmetric matrix

Def: A matrix is called Skew symmetric, if $a_{ij} = - a_{ji}$

i. e. $\mathbf{A^T = -A}$

- ✓ It is worth noting that All the diagonal elements must be zero.
- ✓ Following example is self explanatory

<u>Symmetric</u>

$$\begin{bmatrix} a & h & g \\ h & b & f \\ g & f & c \end{bmatrix}$$

<u>Skew symmetric</u>

$$\begin{bmatrix} 0 & -h & g \\ h & 0 & -f \\ -g & f & 0 \end{bmatrix}$$

<u>Important:</u>

Symmetric Matrix: $\mathbf{A^T = A}$

Skew Symmetric Matrix: $\mathbf{A^T = - A}$

1.3.11. Upper Triangular matrix

Def: A matrix is said to be "upper triangular" if all the elements below its principal diagonal are zeros.

1.3.12. Lower Triangular matrix

Def: A matrix is said to be "lower triangular" if all the elements above its principal diagonal are zeros.

- ✓ Following example is self explanatory

$$\begin{bmatrix} a & h & g \\ 0 & b & f \\ 0 & 0 & c \end{bmatrix}$$
Upper triangular matrix

$$\begin{bmatrix} a & 0 & 0 \\ g & b & 0 \\ f & h & c \end{bmatrix}$$
Lower triangular matrix

1.3.13. Orthogonal matrix:

Def: If $A. A^T = I$, then matrix A is said to be Orthogonal matrix.

1.3.14. Singular matrix:

Def: If $|A| = 0$, then A is called a singular matrix.

For example, $A = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}, B = \begin{bmatrix} 12 & -6 \\ 2 & -1 \end{bmatrix}$

1.3.15. Unitary matrix

Def: If we define, $A^\theta = \overline{(A)}^T$ = transpose of a conjugate of matrix A

Then the matrix is unitary if $A^\theta . A = I$

Let's understand it through an example

If $A = \begin{bmatrix} \frac{1+i}{2} & \frac{-1+i}{2} \\ \frac{1+i}{2} & \frac{1-i}{2} \end{bmatrix}$,

$\Rightarrow A^\theta = \begin{bmatrix} \frac{1-i}{2} & \frac{1-i}{2} \\ \frac{-1-i}{2} & \frac{1+i}{2} \end{bmatrix}$

Since, $A. A^\theta = I$, and Hence it's a Unitary Matrix

1.3.16. Hermitian matrix

Def: It is a square matrix with complex entries which is equal to its own conjugate transpose.

$A^\theta = A$ or $a_{ij} = \overline{a_{ij}}$

For example: $\begin{bmatrix} 5 & 1-i \\ 1+i & 5 \end{bmatrix}$

Note: In a Hermitian matrix, diagonal elements are always real

1.3.17. Skew Hermitian matrix

Def: It is a square matrix with complex entries which is equal to the negative of conjugate transpose.

$A^\theta = -A$ or $a_{ij} = -\overline{a_{ji}}$

For example: $\begin{bmatrix} 0 & 1-i \\ 1+i & 0 \end{bmatrix}$

Note : In Skew-Hermitian matrix , diagonal elements → either zero or Pure Imaginary

1.3.18. Nilpotent Matrix

Def: If $A^k = 0$ (null matrix), then A is called Nilpotent matrix (where K is a +ve integer).

1.3.19. Periodic Matrix

Def: If $A^{k+1} = A$ (where k is a +ve integer), then A is called Periodic matrix.

1.3.20. Idempotent Matrix

Def: If $A^2 = A$, then the matrix A is called idempotent matrix.

For example, $A = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$, $B = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$

1.3.21. Proper Matrix

If $|A| = 1$, matrix A is called Proper Matrix.

1.3.22. Involutory Matrix

Def: If $A^2 = I$, then the matrix A is called involutory matrix.

For example, $A = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$

1.4. Real matrix Vs Complex matrix- classifications

Sl. No.	Classification of Real Matrices	Classification of Complex Matrices
1	Symmetric matrix	Hermitian matrix
2	Skew Symmetric matrix	Skew Hermitian matrix
3	Orthogonal matrix	Unitary matrix

1.5. Equality of matrices

Two matrices can be equal if

- ✓ they are of Same order &
- ✓ Each corresponding element in both the matrices are equal.

1.6. Addition of matrices

Condition: Two matrices can only be added if they are of same size.

Addition of two matrices can be summarized from the following example:

$$\begin{bmatrix} a_1 & b_1 \\ c_1 & d_1 \end{bmatrix} + \begin{bmatrix} a_2 & b_2 \\ c_2 & d_2 \end{bmatrix} = \begin{bmatrix} a_1 + a_2 & b_1 + b_2 \\ c_1 + c_2 & d_1 + d_2 \end{bmatrix}$$

Properties:

1. Addition is commutative
   $e.g.$ A+B = B+A
2. Addition is associative
   $e.g.$ (A+B) +C = A+ (B+C) = B + (C+A)

3. Existence of additive identity: A+O = O+A (Here O is null matrix)
4. If A+P = A+Q, then P = Q

1.7. Subtraction of matrices

Condition: Two matrices can only be subtracted if they are of same size.

Subtraction of two matrices can be summarized from the following example:

$$\begin{bmatrix} a_1 & b_1 \\ c_1 & d_1 \end{bmatrix} - \begin{bmatrix} a_2 & b_2 \\ c_2 & d_2 \end{bmatrix} = \begin{bmatrix} a_1 - a_2 & b_1 - b_2 \\ c_1 - c_2 & d_1 - d_2 \end{bmatrix}$$

Note:

Subtraction is neither commutative & associative
$e.g.$ A-B ≠ B-A

1.8. Multiplication of a matrix by a Scalar:

If a matrix is multiplied by a scalar quality, then each and every element of the matrix gets multiplied by that scalar.

For example:

- ✓ $k\begin{bmatrix} a_1 & b_1 \\ c_1 & d_1 \end{bmatrix} = \begin{bmatrix} ka_1 & kb_1 \\ kc_1 & kd_1 \end{bmatrix}$, Where k is a scalar
- ✓ $k\,m\begin{bmatrix} a_1 & b_1 \\ c_1 & d_1 \end{bmatrix} = m\begin{bmatrix} ka_1 & kb_1 \\ kc_1 & kd_1 \end{bmatrix}$, where k & m are scalar
- ✓ k(X+Y) = kX + kY , Where k is a scalar and X & Y are matrices

1.9. Multiplication of two matrices:

Condition: Two matrices can be multiplied only when number of columns of the first matrix is equal to the number of rows of the second matrix.

Note: Multiplication of $(m \times n)$ and $(n \times p)$ matrices results in matrix of $(m \times p)$ dimension

In simple notation,

$\left[\begin{matrix} m \times \cancel{n} \\ \cancel{n} \times p \end{matrix} = m \times p\right]$.

Example:

$$\begin{bmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{bmatrix} . \begin{bmatrix} m_1 & n_1 \\ m_2 & n_2 \\ m_3 & n_3 \end{bmatrix} = \begin{bmatrix} a_1m_1 + b_1m_2 + c_1m_3 & a_1n_1 + b_1n_2 + c_1n_3 \\ a_2m_1 + b_2m_2 + c_2m_3 & a_2n_1 + b_2n_2 + c_2n_3 \\ a_3m_1 + b_3m_2 + c_3m_3 & a_3n_1 + b_3n_2 + c_3n_3 \end{bmatrix}$$

$(3\times 3) \quad (3\times 2) \quad (3\times 2)$

- AB implies A is Post multiplied by matrix B
- BA implies A is Pre-multiplied by matrix B

1.10. Important properties of matrices

1. 0 A = A 0 = 0, (0 is null matrix)
2. IA = AI = A, (Here A is square matrix of the same order as that of I)
3. If AB = 0, then it is not necessarily that A or B is null matrix. Also at doesn't mean BA = 0

   Example: AB = $\begin{bmatrix} 2 & 2 \\ 2 & 2 \end{bmatrix} . \begin{bmatrix} 2 & -2 \\ -2 & 2 \end{bmatrix} = 0$
4. If the product of two non-zero square matrix A & B is a zero matrix, then A & B are singular matrix.
5. If A is non-singular matrix and A.B=0, then B is null matrix.

   Note: Combining point 3, 4 & 5 it can be said that if product of two square matrices A & B is a zero matrix then either both the matrices are singular matrices or at least one of the matrix is null.

6. Commutative property is not applicable (In general, AB $\neq$ BA)
7. Associative property holds

   i.e. A(BC) = (A B)C
8. Distributive property holds

   i.e. A(B+C) = AB+ AC
9. AC = AD , doesn't imply C = D [even when A $\neq$ 0].
10. If A, C, D be nxn matrix, then if rank (A) = n and AC=AD, then C=D.

Following three demonstrative examples will help in building the concepts.

Examples to demonstrate some of the properties:-

Demonstration 1: Prove by example that AB ≠ BA

Assume, A = $\begin{bmatrix} 1 & 3 \\ -2 & 0 \end{bmatrix}$, B= $\begin{bmatrix} 1 & -4 \\ 2 & 5 \end{bmatrix}$

Then,

AB = $\begin{bmatrix} 15 & -21 \\ -2 & 8 \end{bmatrix}$,

BA = $\begin{bmatrix} 17 & 3 \\ -2 & 8 \end{bmatrix}$

Hence AB ≠ BA

Demonstration 2: Show by an example that AB = 0 $\nRightarrow$ A = 0 or B = 0 or BA = 0

Assume, A & B are two null matrices

A = $\begin{bmatrix} 1 & 1 \\ 2 & 2 \end{bmatrix}$, B= $\begin{bmatrix} -1 & 1 \\ 1 & -1 \end{bmatrix}$

Then,

AB = $\begin{bmatrix} 1 & 1 \\ 2 & 2 \end{bmatrix} . \begin{bmatrix} -1 & 1 \\ 1 & -1 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$

BA = $\begin{bmatrix} 1 & 1 \\ -1 & -1 \end{bmatrix}$

Demonstration 3: Show by an example that AC = AD $\nRightarrow$ C = D (even when A ≠ 0)

Assume, A = $\begin{bmatrix} 1 & 1 \\ 2 & 2 \end{bmatrix}$, C= $\begin{bmatrix} 2 & 1 \\ 2 & 2 \end{bmatrix}$, D = $\begin{bmatrix} 3 & 0 \\ 1 & 3 \end{bmatrix}$

Then,

AC = $\begin{bmatrix} 1 & 1 \\ 2 & 2 \end{bmatrix} . \begin{bmatrix} 2 & 1 \\ 2 & 2 \end{bmatrix} = \begin{bmatrix} 4 & 3 \\ 8 & 6 \end{bmatrix}$

AD = $\begin{bmatrix} 1 & 1 \\ 2 & 2 \end{bmatrix} . \begin{bmatrix} 3 & 0 \\ 1 & 3 \end{bmatrix} = \begin{bmatrix} 4 & 3 \\ 8 & 6 \end{bmatrix}$

This proves that although AC = AD, but C ≠ D

1.11. Determinant

Basically a determinant is nothing but a convenient way of representing a particular number. However we write, it can be reduced to a single number.

Determinants were originally introduced for solving linear systems and have important engineering applications. e.g. electrical networks, frameworks in mechanics, curve fitting and other optimization problems.

An n^{th} order determinant is an expression associated with n × n square matrix.

In general the determinant of "order n" is represented as

$$D = |A| = \det A = \begin{vmatrix} a_{11} & a_{12} & a_{13} & - & - & a_{1n} \\ a_{21} & - & - & - & - & a_{2n} \\ - & & - & - & - & - \\ - & & - & - & - & - \\ a_{n1} & a_{n2} & - & - & - & a_{nn} \end{vmatrix}$$

If $A = [a_{ij}]$, Element a_{ij} with i^{th} row, j^{th} column.

For n = 2 , $D = \det A = \begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix} = a_{11}\,a_{22} - a_{12}\,a_{21}$

1.11.1. Minors

The minor of an element in a determinant is the determinant obtained by deleting the row and the column which intersect that element.

For example, If D is a 3x3 determinant, then

$$D = \begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix}$$

Minor of $a_1 = \begin{vmatrix} b_2 & b_3 \\ c_2 & c_3 \end{vmatrix}$

1.11.2. Co-factors

Cofactor is the minor with "proper sign". The sign is given by $(-1)^{i+j}$ (where the element belongs to i^{th} row, j^{th} column).

A_2 = Cofactor of $a_2 = (-1)^{1+2} \times \begin{vmatrix} b_1 & b_3 \\ c_1 & c_3 \end{vmatrix}$

Cofactor matrix can be formed as $\begin{vmatrix} A_1 & A_2 & A_3 \\ B_1 & B_2 & B_3 \\ C_1 & C_2 & C_3 \end{vmatrix}$

1.11.3. Laplace expansion (for a 3x3 determinant)

$$D = \begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix}$$

$|A| = \det A = \Delta = a_1A_1 + a_2A_2 + a_3A_3 = a_1A_1 + b_1B_1 + c_1C_1$

(In fact we can expand about any row or column)

1.11.4. Properties of Determinants

In a general manner, a row or column is referred as line.

1. A determinant does not change if its rows & columns are interchanged.
   For example,
$$\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} = \begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix}$$

2. Determinant is zero if two parallel lines are identical.

3. If two parallel lines of a determinant are inter-changed, the determinant retains it numerical values but changes in sign.
$$\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} = -\begin{vmatrix} a_1 & c_1 & b_1 \\ a_2 & c_2 & b_2 \\ a_3 & c_3 & b_3 \end{vmatrix} = \begin{vmatrix} c_1 & a_1 & b_1 \\ c_2 & a_2 & b_2 \\ c_3 & a_3 & b_3 \end{vmatrix}$$

4. If each element of a line be multiplied by the same factor, the whole determinant is multiplied by that factor. *[Important to note]*
$$P\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} = \begin{vmatrix} a_1 & Pb_1 & c_1 \\ a_2 & Pb_2 & c_2 \\ a_3 & Pb_3 & c_3 \end{vmatrix}$$

5. If each element of a line consists of the m terms, then determinant can be expressed as sum of the m determinants.
$$\begin{vmatrix} a_1 & b_1 & c_1+d_1-e_1 \\ a_2 & b_2 & c_2+d_2-e_2 \\ a_3 & b_3 & c_3+d_3-e_3 \end{vmatrix} = \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} + \begin{vmatrix} a_1 & b_1 & d_1 \\ a_2 & b_2 & d_2 \\ a_3 & b_3 & d_3 \end{vmatrix} - \begin{vmatrix} a_1 & b_1 & e_1 \\ a_2 & b_2 & e_2 \\ a_3 & b_3 & e_3 \end{vmatrix}$$

6. If each element of a line be added equimultiple of the corresponding elements of one or more parallel lines, determinant is unaffected.
   e.g. By the operation, $R_2 \rightarrow R_2 + pR_1 + qR_3$, determinant is unaffected.

1.11.5. Important points related to determinant:

1. Determinant of an upper triangular/ lower triangular/diagonal/scalar matrix is equal to the product of the principal diagonal elements of the matrix.
2. If A & B are square matrix of the same order, then $|AB|=|BA|=|A||B|$.
3. If A is non singular matrix, then $|A^{-1}|=\frac{1}{|A|}$
4. Determinant of a skew symmetric matrix of odd order is zero.
5. If A is a unitary matrix or orthogonal matrix (i.e. $A^T = A^{-1}$) then $|A| = \pm 1$.

6. **Very Important point:**
   If A is a square matrix of order n, then $|kA| = k^n|A|$.
7. $|I_n| = 1$ (I_n is the identity matrix of order n).
8. The sum of product of the elements of any row (or column) with the cofactors of corresponding elements, is equal to determinant it self.
9. The sum of product of the elements of any row (or column) with the cofactors of some other row (or column), is equal to zero.
   In other words,
   $If\ a_i, b_i, c_i\ are\ the\ matrix\ elements\ and\ A_i, B_i, C_i\ are\ corresponding\ cofactors.$
   - ✓ $a_i A_j + b_i B_j + c_i C_j = \Delta \quad if\ i = j$
   - ✓ $a_i A_j + b_i B_j + c_i C_j = 0 \quad if\ i \neq j$
10. It is worth noting that determinant cannot be expanded about the diagonal. It can only be expanded about a row or column.

1.11.6. Multiplication of determinants

In determinants multiplication, row to row is multiplied (instead of row to column which is done for matrix). The product of two determinants of same order results into a determinant of that order.

1.11.7. Comparison of Determinants & Matrices

A determinant and matrix is totally different thing and it is not correct to even compare them. Following comparative table is made to help students in remembering the concepts related to determinant & matrices.

Sl No	Matrix	Determinant
1	The matrix can't be reduced to one number	The determinant can be reduced to one number
2	*In terms of rows & Columns:* Number of Row and column need not be same (For square matrix, it is equal & for rectangular matrix, it is unequal)	*In terms of rows & Columns:* Number of rows = Number of columns (always)
3	Interchanging rows and columns changes the meaning all together	Interchanging row and column has no effect on the over value of the determinant
4	*Property of Scalar multiplication:* If a matrix is multiplied by a scalar constant, then all elements of matrix is multiplied by that constant.	*Property of Scalar multiplication:* If a determinant is multiplied by a scalar constant, then the elements of one line (i.e. one row or column) is multiplied by that constant.

5	*Property of matrix multiplication:* Multiplication of the 2 matrices is done by multiplying rows of first matrix & column of second matrix	*Property of determinant multiplication:* Multiplication of 2 determinants is done by multiplying rows of first matrix & rows of second matrix

1.11.8. Transpose of Matrix:

Def: The process of interchanging rows & columns is called the transposition of a matrix and denoted by A^T.

Example: $A = \begin{bmatrix} 11 & 12 \\ 15 & 0 \\ 14 & 16 \end{bmatrix}$ Transpose of A= Trans (A)= $A' = A^T = \begin{bmatrix} 11 & 15 & 14 \\ 2 & 0 & 16 \end{bmatrix}$

Note: If A is a square matrix, then A matrix can always be written as sum of symmetric matrix & skew-symmetric matrix

$$A = \frac{1}{2}(A + A^T) + \frac{1}{2}(A - A^T) = \text{symmetric matrix + skew-symmetric matrix}$$

Property of Transpose of Matrix:

- $(A^T)^T = A$
- If A & B are symmetric, then AB+BA is symmetric and AB-BA is skew symmetric.
- If A is symmetric, then A^n is symmetric (n=2, 3, 4…….).
- If A is skew-symmetric, then A^n is symmetric when n is even and skew symmetric when n is odd.
- $(A + B)^T = A^T + B^T$
- $(AB)^T = B^T. A^T$
- $(kA)^T = k.A^T$ (k is scalar, A is vector)
- $(kA)^{-1} = k^{-1}. A^{-1}$ (k is scalar , A is vector)
- $(A^{-1})^T = (A^T)^{-1}$
- $(\overline{A^T}) = (\overline{A})^T$ (Conjugate of a transpose of matrix= Transpose of conjugate of matrix)

1.11.9. Conjugate of Matrix:

For a given matrix P, the matrix obtained by replacing its elements by the corresponding conjugate complex number is called the conjugate of P (and is represented as $\overline{P}$).

Example:

If $P = \begin{bmatrix} 2 + 3i & 8 \\ -i & 7 \end{bmatrix}$,

Then,

$\overline{P} = \begin{bmatrix} 2 - 3i & 8 \\ i & 7 \end{bmatrix}$

1.11.10. Adjoint of a matrix:

Def: Transposed matrix of the cofactors of A is called the adjoint of a matrix.

Notionally,

Adj (A)= Trans (cofactor matrix)

Lets assume a square matrix A = $\begin{bmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{bmatrix}$

Its Determinant is $\Delta = \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}$

The matrix formed by the cofactors of the elements in Δ is called co factor matrix.

Cofactor matrix = $\begin{bmatrix} A_1 & B_1 & C_1 \\ A_2 & B_2 & C_2 \\ A_3 & B_3 & C_3 \end{bmatrix}$

Taking the transpose of the above matrix, we get Adj (A)

Adj (A) = $\begin{bmatrix} A_1 & A_2 & A_3 \\ B_1 & B_2 & B_3 \\ C_1 & C_2 & C_3 \end{bmatrix}$

1.11.11. Inverse of a matrix

Def: The inverse of a matrix A, if A is non-singular is defined as

$A^{-1} = \frac{\text{Adj } A}{|A|}$

- Inverse exists only if $|A|$ must be non-zero.
- Inverse of a matrix, if exists, is always unique.
- **Short Cut formula (important)**

  If it is a 2x2 matrix $\begin{bmatrix} a & b \\ c & d \end{bmatrix}$, its inverse will be $\frac{1}{ad-bc}\begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$

Important Points

1. $(AB)^{-1} = B^{-1}.\ A^{-1}$
2. $A\,A^{-1} = A^{-1}A = I$
3. If a non-singular matrix A is symmetric, then A^{-1} is also symmetric.
4. If A is a orthogonal matrix , then A^T and A^{-1} are also orthogonal.
5. If A is a square matrix of order n then (i) $|\text{adj } A| = |A|^{n-1}$

   (ii) $|\text{adj } (\text{adj } A)| = |A|^{(n-1)^2}$

   (iii) $\text{adj } (\text{adj } A) = |A|^{n-2}A$

Example: Prove $(A\,B)^{-1} = B^{-1}\ A^{-1}$

Proof: RHS = $(B^{-1} A^{-1})$

Pre-multiplying the RHS by AB, $(A B)(B^{-1} A^{-1}) = A(B.B^{-1})A^{-1} = I$

Similarly, Post-multiplying the RHS by AB, $(B^{-1} A^{-1})(A B) = B^{-1}(A A^{-1})B = B^{-1}B = I$

Hence, AB & B^{-1} A^{-1} are inverse to each other

Example: Symmetric and skew symmetric matrix of the following matrix A is

$$A = \begin{bmatrix} 1 & 2 & 4 \\ -2 & 5 & 3 \\ -1 & 6 & 3 \end{bmatrix}$$

Solution: Symmetric matrix = $\frac{1}{2}(A + A^T) = \frac{1}{2}\left\{\begin{bmatrix} 1 & 2 & 4 \\ -2 & 5 & 3 \\ -1 & 6 & 3 \end{bmatrix} + \begin{bmatrix} 1 & -2 & -1 \\ 2 & 5 & 6 \\ 4 & 3 & 3 \end{bmatrix}\right\}$

$$= \frac{1}{2}\begin{bmatrix} 2 & 0 & 3 \\ 0 & 10 & 9 \\ 3 & 9 & 6 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 3/2 \\ 0 & 5 & 9/2 \\ 3/2 & 9/2 & 3 \end{bmatrix}$$

Skew symmetric matrix = $\frac{1}{2}(A - A^T) = \begin{bmatrix} 0 & 2 & 5/2 \\ -2 & 0 & -3/2 \\ -5/2 & 3/2 & 0 \end{bmatrix}$

Example: Whether the following matrix A is orthogonal?

$$A = \frac{1}{3}\begin{bmatrix} 1 & 2 & 2 \\ 2 & 1 & -2 \\ -2 & 2 & -1 \end{bmatrix}$$

Solution: We know that if $A . A^T = I$, then the matrix is orthogonal.

$$A' = A^T = \frac{1}{3}\begin{bmatrix} 1 & 2 & -2 \\ 2 & 1 & 2 \\ 2 & -2 & -1 \end{bmatrix}$$

$$A . A^T = A . A' = \frac{1}{9}\begin{bmatrix} 9 & 0 & 0 \\ 0 & 9 & 0 \\ 0 & 0 & 9 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = I,$$

Hence A is orthogonal matrix.

Example: Find A for the following equation

$$\begin{bmatrix} 2 & 1 \\ 3 & 2 \end{bmatrix} A \begin{bmatrix} -3 & 2 \\ 5 & -3 \end{bmatrix} = \begin{bmatrix} -2 & 4 \\ 3 & -1 \end{bmatrix}$$

Solution:

We, if it is a 2x2 matrix $\begin{bmatrix} a & b \\ c & d \end{bmatrix}$, its inverse will be $\frac{1}{ad-bc}\begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$

Pre multiply both side of matrix by inverse of $\begin{bmatrix} 2 & 1 \\ 3 & 2 \end{bmatrix}$ i.e. $\begin{bmatrix} 2 & -1 \\ -3 & 2 \end{bmatrix}$

$$\begin{bmatrix} 2 & -1 \\ -3 & 2 \end{bmatrix}\begin{bmatrix} 2 & 1 \\ 3 & 2 \end{bmatrix} A \begin{bmatrix} -3 & 2 \\ 5 & -3 \end{bmatrix} = \begin{bmatrix} +2 & -1 \\ -3 & 2 \end{bmatrix}\begin{bmatrix} -2 & 4 \\ 3 & -1 \end{bmatrix}$$

$$A\begin{bmatrix} -3 & 2 \\ 5 & -3 \end{bmatrix} = \begin{bmatrix} -7 & 9 \\ 12 & -14 \end{bmatrix}$$

Post multiply both side of matrix by inverse of $\begin{bmatrix} -3 & 2 \\ 5 & -3 \end{bmatrix}$ = (-) $\begin{bmatrix} -3 & -2 \\ -5 & -3 \end{bmatrix}$= $\begin{bmatrix} 3 & 2 \\ 5 & 3 \end{bmatrix}$

$$A = \begin{bmatrix} -3 & 2 \\ 5 & -3 \end{bmatrix}\begin{bmatrix} +3 & +2 \\ +5 & +3 \end{bmatrix} = \begin{bmatrix} -7 & 9 \\ 12 & -14 \end{bmatrix}\begin{bmatrix} +3 & +2 \\ +5 & +3 \end{bmatrix}$$

$$A\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 24 & 13 \\ -34 & -18 \end{bmatrix}$$

$$A = \begin{bmatrix} 24 & 13 \\ -34 & -18 \end{bmatrix}$$

1.12. Elementary transformation of matrix:

Following three operations forms the basis of elementary transformation

1. Interchange of any 2 lines
2. Multiplication of a line by a constant (e.g. $k\,R_i$)
3. Addition of constant multiplication of any line to the another line (e. g. $R_i + p\,R_J$)

Important point:

- A linear system S1 is called **"Row Equivalent"** to linear system S2, if S1 can be obtained from S2 by finite number of elementary row operations.
- Elementary transformations don't change the rank of the matrix. However it changes the Eigen value of the matrix. (Plz check with an example).

1.13. Gauss-Jordan method of finding Inverse

Process: Elementary row transformations which reduces a given square matrix A to the unit matrix, when applied to unit matrix I, gives the inverse of A.

Example: Find the inverse of matrix $\begin{bmatrix} -1 & 1 & 2 \\ 3 & -1 & 1 \\ -1 & 3 & 4 \end{bmatrix}$

Solution: Writing in the form, such that Identity matrix is appended at the end

$$\begin{bmatrix} -1 & 1 & 2 & : & 1 & 0 & 0 \\ 3 & -1 & 1 & : & 0 & 1 & 0 \\ -1 & 3 & 4 & : & 0 & 0 & 1 \end{bmatrix}$$

Using elementary transformation, $R_2 \to R_2 + 3R_1$, $R_3 \to R_3 - R_1$

$$\begin{bmatrix} -1 & 1 & 2 & : & 1 & 0 & 0 \\ 0 & 2 & 7 & : & 3 & 1 & 0 \\ 0 & 2 & 2 & : & -1 & 0 & 1 \end{bmatrix}$$

Using elementary transformation, $R_3 \to R_3 - R_2$

$$\begin{bmatrix} -1 & 1 & 2 & : & 1 & 0 & 0 \\ 0 & 2 & 7 & : & 3 & 1 & 0 \\ 0 & 0 & -5 & : & -4 & -1 & 1 \end{bmatrix}$$

Using elementary transformation, $R_1 \to -R_1$, $R_2 \to R_2/2$, $R_3 \to -R_3/5$

$$\begin{bmatrix} 1 & -1 & -2 & : & -1 & 0 & 0 \\ 0 & 1 & 7/2 & : & 3/2 & 1/2 & 0 \\ 0 & 0 & 1 & : & 4/5 & 1/5 & -1/5 \end{bmatrix}$$

Using elementary transformation, $R_2 \to R_2 - \frac{7\ R_3}{2}$, $R_1 \to R_1 + 2R_3$

$$\begin{bmatrix} 1 & -1 & 0 & : & 3/5 & 2/5 & -2/5 \\ 0 & 1 & 0 & : & -13/10 & -1/5 & 7/10 \\ 0 & 0 & 1 & : & 4/5 & 1/5 & -1/5 \end{bmatrix}$$

Using elementary transformation, $R_1 \to R_1 + R_2$

$$\begin{bmatrix} 1 & 0 & 0 & : & -7/10 & 1/5 & 3/10 \\ 0 & 1 & 0 & : & -13/10 & -1/5 & 7/10 \\ 0 & 0 & 1 & : & 4/5 & 1/5 & -1/5 \end{bmatrix}$$

1.14. Rank of matrix (important)

Definition of minor:

If we select any r rows and r columns from any matrix A, deleting all other rows and columns, then the determinant formed by these $r \times r$ elements is called minor of A of order r.

Definition of rank:

A matrix is said to be of rank r when,

i) It has at least one non-zero minor of order r.
ii) Every minor of order higher than r vanishes.

Other definition: The rank is also defined as maximum number of linearly independent row vectors.

Special case: Rank of Square matrix

Using elementary transformation, convert the matrix into the upper triangular matrix. Number of non-zero row gives the rank.

Note:

1. For two matrices A & B, $r(A.B) \leq \min\{r(A), r(B)\}$
2. For two matrices A & B, $r(A+B) \leq r(A) + r(B)$
3. For two matrices A & B, $r(A-B) \geq r(A) - r(B)$
4. The rank of a diagonal matrix is simply the number of non-zero elements in principal diagonal.
5. For a matrix A, r(A)=0 iff A is a null matrix.
6. If two matrices A and B have the same size and the same rank then A, B are equivalent matrices.
7. Every non-singular matrix is row equivalent to identity matrix.
8. A system of homogeneous equations such that the number of unknown variable exceeds the number of equations, necessarily has non-zero solutions.
9. If A is a non-singular matrix, then all the row/column vectors are independent.
10. If A is a singular matrix, then vectors of A are linearly dependent.

Example: Let a matrix is given as A= $\begin{bmatrix} \beta & -1 & 0 \\ 0 & \beta & -1 \\ -1 & 0 & \beta \end{bmatrix}$. Also it is known that its rank is 2 , then find the value of β.

Solution:

Its rank is 2, hence its determinant is equal to zero.

$$\beta^3 - 1 = 0$$

$$\beta = 1$$

Example: What is the rank of matrix A= $\begin{bmatrix} 2 & 3 & -1 & -1 \\ 1 & -1 & -2 & -4 \\ 3 & 1 & 3 & -2 \\ 6 & 3 & 0 & -7 \end{bmatrix}$?

Solution: Applying the elementary transformation,

$$R_1 \leftrightarrow R_2, \qquad R_2 - 2R_1, R_3 - 3R_1, R_4 - 6R_1,$$

$$\sim \begin{bmatrix} 1 & -1 & -2 & -4 \\ 2 & 3 & -1 & -1 \\ 3 & 1 & 3 & -2 \\ 6 & 3 & 0 & -7 \end{bmatrix} \sim \begin{bmatrix} 1 & -1 & -2 & -4 \\ 0 & 5 & 3 & 7 \\ 0 & 4 & 9 & 10 \\ 0 & 9 & 12 & 17 \end{bmatrix}$$

By elementary transformation,

$$R_3 - \frac{4}{5}R_2,\ R_4 - \frac{9}{5}R_2 \qquad\qquad R_4 - R_3$$

$$\sim \begin{bmatrix} 1 & -1 & -2 & -4 \\ 0 & 5 & 3 & 7 \\ 0 & 0 & 33/5 & 22/5 \\ 0 & 0 & 33/5 & 22/5 \end{bmatrix} \sim \begin{bmatrix} 1 & -1 & -2 & -4 \\ 0 & 5 & 3 & 7 \\ 0 & 0 & 33/5 & 22/5 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

Number of Non zero rows is 3 and Hence Rank = 3

Example: What is the rank of a diagonal matrix $\begin{bmatrix} -1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 4 \end{bmatrix}$?

Solution: It is a diagonal matrix and we know that the number of non-zero elements in the diagonal matrix gives the rank. Hence its rank is 3.

Example: What is the rank of matrix A= $\begin{bmatrix} 2 & -4 & 6 \\ -1 & 2 & -3 \\ 3 & -6 & 9 \end{bmatrix}$?

Solution:

Applying the elementary transformation,

$R_3 \to R_3 + 3R_2$ *and* $R_1 \to R_1 + 2R_2$, we obtain $\begin{bmatrix} 0 & 0 & 0 \\ -1 & 2 & -3 \\ 0 & 0 & 0 \end{bmatrix}$

Hence rank is 1 (As number of non-zero row is 1.).

Example: Find rank of A = $\begin{bmatrix} 3 & 0 & 2 & 2 \\ -6 & 42 & 24 & 54 \\ 21 & -21 & 0 & -15 \end{bmatrix}$

Solution: $R_2 \to R_2 + 2R_1,\ R_3 \to R_3 - 7R_1,$ $\begin{bmatrix} 3 & 0 & 2 & 2 \\ 0 & 42 & 28 & 58 \\ 0 & -21 & -14 & -29 \end{bmatrix}$

$R_2 \to R_2 + 2R_3,$ $\begin{bmatrix} 3 & 0 & 2 & 2 \\ 0 & 0 & 0 & 0 \\ 0 & -21 & -14 & -29 \end{bmatrix}$

There are two non-zero rows and hence Rank of A is 2

1.15. Some important definitions:

Vector space

It is a set V of non-empty vectors such that with any two vectors "a and b" in V, all their linear combinations $\alpha a + \beta b$ (α, β are real numbers) are elements of V.

Dimension

The maximum number of linearly independent vectors in V is called the dimension of V (and denoted as dim V).

Basis

A linearly independent set in V consisting of a maximum possible number of vectors in V is called a basis for V. Thus the number of vectors of a basis for V is equal to dim V.

Span

The set of all linear combinations of given vector $a_{(1)}, a_{(2)}, \ldots\ldots\ldots\ldots a_{(p)}$ with same number of components is called the span of these vectors. Obviously, a span is a vector space.

Vector:

Def: Any quantity having n components is called a vector of order n.

1.16. Linear Dependence of Vectors

If one vector can be written as linear combination of others, the vector is linearly dependent.

1.17. Linearly Independent Vectors

If no vectors can be written as a linear combination of others, then they are linearly independent.

Example: Suppose the vectors are x_1 x_2 x_3 x_4 x_5

Its linear combination is $\lambda_1 x_1 + \lambda_2 x_2 + \lambda_3 x_3 + \lambda_4 x_4 + \lambda_5 x_5 = 0$

- If $\lambda_1, \lambda_2, \lambda_3, \lambda_4, \lambda_5$ are "not all zero", then they are linearly dependent.
- If "all λ" are zero → they are linearly independent.

1.18. System of linear System of Equation

Any set of values simultaneously satisfying all the equations is called the solution of the system of equations.

- The equations are said to be Consistent if they posses one or more solution exists (i.e. unique or infinite solution)
- The equations are said to be inconsistent if they posses no solution exists

Lets assume, following system of equations are given

$$\begin{array}{l} a_{11}x_1 + a_{12}x_2 + - \quad - \quad - \quad a_{1n}x_n = k_1 \\ a_{21}x_1 + a_{22}x_2 + - \quad - \quad - \quad a_{2n}x_n = k_2 \\ - \quad - \quad - \quad - \quad - \quad - \quad - \\ - \quad - \quad - \quad - \quad - \quad - \quad - \\ a_{m1}x_1 + a_{m2}x_2 + - \quad - \quad - \quad a_{mn}x_n = k_m \end{array}$$

If we write,

$$A = \begin{bmatrix} a_{11} & a_{12} & - & - & - & a_{1n} \\ a_{21} & - & - & - & - & a_{2n} \\ - & - & - & - & - & - \\ - & - & - & - & - & - \\ - & - & - & - & - & \\ a_{m1} & a_{m2} & - & - & - & a_{mn} \end{bmatrix},$$

$$X = \begin{bmatrix} x_1 \\ x_2 \\ - \\ - \\ - \\ x_n \end{bmatrix}$$

$$B = \begin{bmatrix} k_1 \\ k_2 \\ - \\ - \\ - \\ k_m \end{bmatrix}$$

In matrix form, it can be written as $AX = B$

A= Coefficient Matrix

C = (A, B) = Augmented Matrix

Further in terms of the symbol (used in standard text books),

r = rank (A), r' = rank (C) , n = Number of unknown variables (x_1 , x_2 , - - - x_n)

1.18.1. Meaning of consistency, inconsistency of Linear Equation

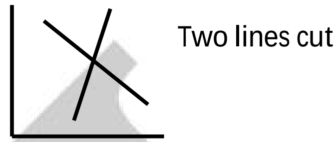
Consistent implies that one or more solution exists (i.e. unique or infinite solution)

Consistent →

Case 1: Unique solution

$2x+3y = 9$

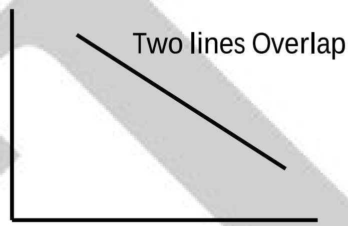
$3x+4y = 12$



Case 2: Infinite solution

$x+3y = 5$

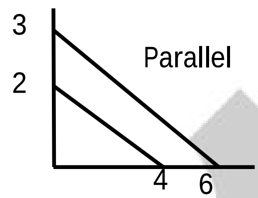
$3x+9y = 15$



Inconsistent implies that no solution exists

$3x+6y = 12$

$x+2y = 6$



1.18.2. Consistency of a system of equations (Important)

Consistency of a system of equations can easily be found using the concept of rank. That's why knowing the concept of rank and means of finding it is utmost important for this topic.

We will discuss the following two Cases:

- **Non-homogenous equations (**A X = B)
- **Homogenous equations (**A X = 0)

Also lets assume,

r = rank (Coefficient Matrix), r' = rank (Augmented Matrix), n = Number of unknown variables

CASE A : For non-homogenous equations (A X = B)

i) If $r \neq r'$, the equations are inconsistent (hence there is no solution).
ii)If $r = r' = n$, the equations are consistent and has unique solution.
iii) If $r = r' < n$, the equations are consistent and has infinite number of solutions.

CASE B : For homogenous equations (A X = 0)

i) If r =n, the equations have only a trivial zero solution (i.e. $x_1 = x_2 = - - - x_n = 0$).
ii) If r<n, then (n-r) linearly independent solution (i.e. infinite non-trivial solutions).

Trivial solution implies obvious solution, such as in homogenous equations (A X = 0) case, all the variable being zero, obviously satisfies the set of equations (No computation is required for this).

Note: Two points to be remembers for finding the consistency

- Finding the rank (through Gauss elimination method + elementary transformation or any other traditional means)
- Remembering the above conditions relating to rank of Coefficient Matrix & Augmented Matrix

1.18.3. Cramer's Rule

At this junction, we would like to introduce you to Cramer's rule. Observe closely the resemblance with finding the Consistency of a system of equations (using rank concept) and using the Cramer's rule concept.

Let the following two equations be there

$$a_{11}\, x_1 + a_{12}\, x_2 = b_1 \text{ ---------------------------------------(i)}$$
$$a_{21}\, x_1 + a_{22}\, x_2 = b_2 \text{ ---------------------------------------(ii)}$$

$$D = \begin{vmatrix} a_{11} & a_{12} \\ b_{21} & b_{22} \end{vmatrix}$$

$$D_1 = \begin{vmatrix} b_1 & a_{12} \\ b_2 & a_{22} \end{vmatrix}$$

$$D_2 = \begin{vmatrix} a_{11} & b_1 \\ a_{21} & b_2 \end{vmatrix}$$

Solution using Cramer's rule:

$x_1 = \frac{D_1}{D}$ and $x_2 = \frac{D_2}{D}$

In the above method, it is assumed that

1. No of equation = no of unknown
2. $D \neq 0$

CASEA: For non-homogenous Equations

If $D \neq 0 \rightarrow$ single solution (non trivial)

If $D = 0 \rightarrow$ infinite solution

CASE B: For homogenous Equations

If $D \neq 0 \rightarrow$ trivial solutions ($x_1 = x_2 = \ldots\ldots\ldots\ldots x_n = 0$)

If $D = 0 \rightarrow$ non- trivial solution (or infinite solution)

Note: Cramer's rule is considered inefficient in terms of computation.

Example: Find the solution of following Simultaneous Equation:

$$x_1 + 2x_2 - x_3 = 1$$
$$3x_1 - 2x_2 + 2x_3 = 2$$
$$7x_1 - 2x_2 + 3x_3 = 5$$

Solution:

It is non homogeneous equation.

The equation can be written as

$$\begin{bmatrix}1 & 2 & -1\\ 3 & -2 & 2\\ 7 & -2 & 3\end{bmatrix}\begin{bmatrix}x_1\\ x_2\\ x_3\end{bmatrix} = \begin{bmatrix}1\\ 2\\ 5\end{bmatrix}$$

Applying the elementary transformation

$R_2 - 3R_1, \quad R_3 - (R_1 + 2R_2)$

$$\begin{bmatrix}1 & 2 & -1\\ 0 & -8 & 5\\ 0 & 0 & 0\end{bmatrix}\begin{bmatrix}x_1\\ x_2\\ x_3\end{bmatrix} = \begin{bmatrix}1\\ -1\\ 0\end{bmatrix}$$

Writing the above matrix notation, in terms of simple equation

$x_1 + 2x_2 - x_3 = 1$ --(i)

$8x_2 + 5x_3 = -1$--(ii)

Assume, $x_3 = k$---(iii)

From (ii), $x_2 = \frac{1}{8}(5k+1)$

From (i), $x_1 = 1 - \frac{1}{4}(5k+1)+k \quad = \frac{1}{4}(3-k)$

For each & every k, there will be a solution set.

Hence the equations have infinite solution.

Example: For the given simultaneous equation, find the value of λ & μ for the following cases:

(A) no solution (B) unique solution (C) infinite solution

matrix is given as,

$$\begin{bmatrix}1 & 1 & 1\\ 1 & 2 & 3\\ 1 & 2 & \lambda\end{bmatrix}\begin{bmatrix}x\\ y\\ z\end{bmatrix} = \begin{bmatrix}6\\ 10\\ \mu\end{bmatrix},$$

Solution: It is given

$$\begin{bmatrix}1 & 1 & 1\\ 1 & 2 & 3\\ 1 & 2 & \lambda\end{bmatrix}\begin{bmatrix}x\\ y\\ z\end{bmatrix} = \begin{bmatrix}6\\ 10\\ \mu\end{bmatrix}$$

$$A \quad X \quad = \quad B$$

$$\text{Augmented matrix} = C = (A, B) = \begin{bmatrix}1 & 1 & 1 & : & 6\\ 1 & 2 & 3 & : & 10\\ 1 & 2 & \lambda & : & \mu\end{bmatrix}$$

Applying elementary transformation $R_2 - R_1$, $R_3 - R_1$

$$\begin{bmatrix} 1 & 1 & 1 & 6 \\ 0 & 1 & 2 & 4 \\ 0 & 1 & l-1 & \mu-6 \end{bmatrix}$$

Applying elementary transformation $R_3 - R_2$

$$\begin{bmatrix} 1 & 1 & 1 & 6 \\ 0 & 1 & 2 & 4 \\ 0 & 0 & l-3 & \mu-10 \end{bmatrix}$$

Taking the above matrix as reference and correlating with the concept of rank, we can get the following results.

A) For no solution : $R(A) \neq R(C)$ i.e, $l - 3 = 0$, but $\mu - 10 \neq 0$

$l = 3, \quad \mu \neq 10$

B) For unique solution: $R(A) = R(C) = 3$

$l - 3 \neq 0$, μ may be anything

$l \neq 3$, μ may be anything

C) Infinite solution: $R(A) = R(C) = 2$

$l - 3 = 0, \quad \mu - 10 = 0$

$l = 3, \quad \mu = 10$

Example: Examine l, μ for the below cases (A), (B) & (C) for the following equations

$$\begin{bmatrix} 2 & 3 & 5 \\ 7 & 3 & -2 \\ 2 & 3 & l \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 9 \\ 8 \\ \mu \end{bmatrix}$$

(A) no solution (B) unique solution (C) infinite solution

Solution: $C = (A, B) = \begin{bmatrix} 2 & 3 & 5 & 9 \\ 7 & 3 & -2 & 8 \\ 2 & 3 & l & \mu \end{bmatrix}$

$R_2 - \frac{7}{2}R_1$, $R_3 - R_1$, $\begin{bmatrix} 2 & 3 & 5 & 9 \\ 0 & -\frac{15}{2} & -2-\frac{35}{2} & 8-\frac{63}{2} \\ 0 & 0 & l-5 & \mu-9 \end{bmatrix}$

A. For no solution : $R(A) \neq R(C)$

$l - 5 = 0, \quad \mu - 9 \neq 0$

$l = 5, \quad \mu \neq 9$

B. For unique solution: $R(A) = R(C) = 3$

$l - 5 \neq 0$, μ can be anything

$l \neq 5$, μ may be anything

C. Infinite solution: R(A) = R (C) = 2

$l - 5 = 0, \quad \mu - 9 = 0$

$l = 5, \quad \mu = 9$

1.19. Characteristic equation and Eigen Values (Important)

Def: If A is a matrix, then the equation $|A - \lambda I| = 0$ is called characteristic equation. Here λ is scalar & I is identity matrix.

- The roots of this equation are called the characteristic roots/ characteristic values/ latent roots/ Eigen values/ proper values of the matrix A.
- By and large Eigen value is the most popular name and perhaps the most important concept of the Linear algebra
- The matrix $[A - \lambda I]$ is called the characteristic matrix of A.

1.20. Eigen vectors (Important)

Lets assume the following equation:

$[A - \lambda I] X = 0$

In other words, we can also write as

$AX = \lambda X$

Solving for X (for each Eigen value λ,) gives the Eigen vectors/ characteristic vector.

Clearly, null/ zero vector (X = 0) is one of the solutions (But it is of no practical interest).

- The beauty related to Eigen value & Eigen vector is that by multiplying the Eigen vector X by a scalar (λ) and matrix (A) gives the same result.
- For a given Eigen value, there can be different Eigen vectors, but for same Eigen vector, there can't be different Eigen values.

Note:

1. The set of Eigen values is called SPECTRUM of A.
2. The largest of the absolute value of Eigen values is called spectral radius of A.

1.21. Properties of Eigen value (Important)

1. The sum of the Eigen values of a matrix is equal to the sum of its principal diagonal (i.e. trace of the matrix).
2. The product of the Eigen values of a matrix is equal to its determinant.
3. If λ is an Eigen value of orthogonal matrix, then $1/\lambda$ is also its Eigen value.
4. If A is real, then its Eigen value is real or complex conjugate pair.
5. The Eigen values of triangular matrix are just the diagonal elements of the matrix.

6. Eigen values of a unitary matrix or orthogonal matrix has absolute value '1'.
7. Eigen values of Hermitian or symmetric matrix are purely real.
8. Eigen values of skew Hermitian or skew symmetric matrix is zero or pure imaginary.
9. If λ is an Eigen value of the matrix then ,
   i) Eigen value of A^m is λ^m
   ii) Eigen value of A^{-1} is $1/\lambda$
   iii) Eigen value of kA are $k\lambda$ (k is scalar)
   iv) Eigen value of $A + kI$ are $\lambda + k$
   v) Eigen value of $(A - kI)^2$ are $(l-k)^2$

   Note: Above properties are very important. Almost always one question is asked based on these properties.
10. The largest Eigen values of a matrix is always greater than or equal to any of the diagonal elements of the matrix.
11. Zero is the Eigen value of the matrix if and only if the matrix is singular.
    Matrix A and its transpose A^T has same characteristic root (Eigen values).
12. If A & B are two matrices of same order, then the matrix AB & Matrix BA will have same Eigen Values.
13. Similar matrices have same Eigen values. Two matrices A & B are said to be similar, if there exists a non singular matrix P such that $B = P^{-1}AP$

1.22. Properties of Eigen Vector

1) Eigen vector X of matrix A is not unique. Actually there are infinite Eigen vectors.
   Let X_i is Eigen vector, then CX_i is also Eigen vector (C = scalar constant).
2) If Eigen values $\lambda_1, \lambda_2, \lambda_3 \ldots \ldots \lambda_n$ are distinct, then $X_1, X_2 \ldots \ldots X_n$ are linearly independent. However, if two or more Eigen values are equal, it may or may not be possible to get linearly independent Eigen vectors corresponding to equal roots.
3) Two Eigen vectors are called orthogonal vectors if $X_1^T \cdot X_2 = 0$.
   (X_1, X_2 are column vector)
   (Note the difference, for a single vector to be orthogonal , A. A^T = A. A^{-1} = I)
4) Eigen vectors of a symmetric matrix corresponding to different Eigen values are orthogonal.

Example: Find Eigen value and Eigen vector of $A = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$

Solution: By definition , $|A - \lambda I| = 0$

$$\begin{vmatrix} -\lambda & 1 \\ -1 & -\lambda \end{vmatrix} = \lambda^2 + 1 = 0,$$

$$\lambda = \pm i$$

i.e. $\lambda_1 = i$, $\lambda_2 = -i$

Eigen vector is given by $[A - \lambda I] X = 0$

For $\lambda_1 = i$, $\begin{bmatrix} -i & 1 \\ -1 & -i \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = 0$

$-ix_1 + x_2 = 0$ ---------------------------------------(i)

$-x_1 - ix_2 = 0$ ---(ii)

Both the equation are same, it can be written as $\frac{x_1}{1} = \frac{x_2}{i}$

Hence Eigen vector is, $\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 1 \\ i \end{bmatrix}$

For $\lambda_2 = -i$, $\begin{bmatrix} i & 1 \\ -1 & i \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = 0$

$ix_1 + x_2 = 0$ ---------------------------------------(i)

$-x_1 + ix_2 = 0$ ---------------------------------------(ii)

Both equations are same, hence it can be written as $\frac{x_1}{1} = \frac{x_2}{-i}$

Hence Eigen vector is, $\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 1 \\ -i \end{bmatrix}$

Note: Real matrices can have complex Eigen value and Eigen vector.

Example: What is the Eigen value and the Eigen vector of $A = \begin{bmatrix} 5 & 4 \\ 1 & 2 \end{bmatrix}$

Solution:

Characteristic Equation is given as, $|A - \lambda I| = 0$

Solving, $\begin{vmatrix} 5-\lambda & 4 \\ 1 & 2-\lambda \end{vmatrix} = 0$

$\lambda^2 - 7\lambda + 6 = 0 \quad \Rightarrow \lambda = 6, 1$

We know Eigen vector is defined as, $[A - \lambda I] X = 0$

Hence, $\begin{bmatrix} 5-\lambda_1 & 4 \\ 1 & 2-\lambda_1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = 0$

For $\lambda = 6$, $\begin{bmatrix} -1 & 4 \\ 1 & -4 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = 0$

$-x_1 + 4x_2 = 0$ ---------------(i)

$x_1 - 4x_2 = 0$ ---------------(ii)

Although there are two equations, but only one equation is independent (because rank is 1)

$\frac{x_1}{4} = \frac{x_2}{1}$ giving the Eigen vector (4, 1)

For $\lambda = 1$, $\begin{bmatrix} 4 & 4 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = 0$

$4x_1 + 4x_2 = 0$ ---------------(iii)

$x_1 + x_2 = 0$ ---------------(iv)

Writing in the standard proportionality form,

We get, $\frac{x_1}{1} = \frac{x_2}{-1}$

Which gives the Eigen vector (1, -1)

1.23. Cayley Hamilton Theorem (Important)

Cayley Hamilton Theorem states that every square matrix satisfies its own characteristic equation.

Application: This theorem provides an alternate way of finding the inverse of a matrix.

Example:

If $\lambda^2 - 14\lambda - 15 = 0$ ---------------(i) is the characteristic equation of a matrix A,

then from Cayley Hamilton Theorem,

$A^2 - 14A - 15I = 0$ ---------------(ii)

Note: Can you guess why I (i.e. Identity matrix) is added in the second equation, when there is no I in the first equation?

1.24. Similar matrices

Two matrices A & B are said to be similar, if there exists a non-singular matrix P such that $B = P^{-1} A P$

Note: Similar matrices have same Eigen values.

1.25. Diagonalisation of a matrix

The process of finding a matrix M such that $M^{-1} A M = D$ (where D is a diagonal matrix) is called Diagonalisation of matrix A.

Observation:

- As A & D are similar matrices and hence they have the same Eigen value.
- Eigen Value of Diagonal matrix is its diagonal elements.

Calculating M:

- First calculate the Eigen value of A
- M is the matrix whose columns are Eigen Vectors of A

Many a time, Diagonalisation is used to calculate the power of matrix.

1.26. Calculating power of a matrix using Diagonalisation:

Problem: Our aim to find the n^{th} power of A $(i.e. A^n)$

Step 1: Calculate M (Using above procedure)

Step 2: Calculate D

$$D = M^{-1} A M$$

Step 3: Calculate A^n

$$A^n = M^{-1} D^n M$$

Observation: n times matrix multiplication is reduced to just 2 multiplication

Example: If A = $\begin{bmatrix} 1 & 2 & 2 \\ 2 & 1 & 2 \\ 2 & 2 & 1 \end{bmatrix}$, find the value of $A^2 - 4A - 5I$

Solution:

$|A - \lambda I| = 0 \Rightarrow -\lambda^3 + 9\lambda + 3\lambda^2 + 5$

$= (-\lambda^2 + 4\lambda + 5)\ (\lambda+1) = 0$

$\lambda^2 - 4\lambda - 5 = 0$ or $\lambda + 1 = 0$

From Cayley Hamilton Theorem, $A^2 - 4A - 5 = 0$ or $A+I=0$

As $A \neq -I$, Hence $A^2 - 4A - 5 = 0$

Example: For matrix A = $\begin{bmatrix} 1 & 2 & -3 \\ 0 & 3 & 2 \\ 0 & 0 & -2 \end{bmatrix}$, find Eigen vector of $3A^3 + 5A^2 - 6A + 2I$.

Solution: $|A - l I| = 0$

$$\begin{vmatrix} 1-l & 2 & -3 \\ 0 & 3-l & 2 \\ 0 & 0 & -2-l \end{vmatrix} = 0$$

$(1-\lambda)(3-\lambda)(-2-\lambda) = 0 \implies \lambda = 1, \lambda = 3, \lambda = -2$

Eigen value of A = 1, 3, -2

Eigen value of A^3 = 1, 27, -8

Eigen value of A^2 = 1, 9, 4

Eigen value of I = 1, 1, 1

First Eigen value of $3A^3 + 5A^2 - 6A + 2I = 3 \times 1 + 5 \times 1 - 6 \times 1 + 2 = 4$

Second Eigen value of $3A^3 + 5A^2 - 6A + 2I = 3(27) + 5(9) - 6(3) + 2 = 81 + 45 - 18 + 2$

$= 110$

Third Eigen value of $3A^3 + 5A^2 - 6A + 2I = 3(-8) + 5(4) - 6(-2) + 2 = -24 + 20 + 12 + 2 = 10$

Example: Find Eigen values of matrix

$$A = \begin{bmatrix} a_{11} & 0 & 0 & 0 \\ a_{21} & a_{22} & 0 & 0 \\ a_{31} & a_{32} & a_{33} & 0 \\ a_{41} & a_{42} & a_{42} & a_{44} \end{bmatrix}$$

Solution:

$$|A-\lambda I| = \begin{vmatrix} a_{11-l} & 0 & 0 & 0 \\ a_{21} & a_{22-l} & 0 & 0 \\ a_{31} & a_{32} & a_{33-l} & 0 \\ a_{41} & a_{42} & a_{42} & a_{44-l} \end{vmatrix} = 0$$

Expanding, $(a_{11-l})\ (a_{22-l})\ (a_{33-l})\ (a_{44-l}) = 0$

$\lambda = a_{11}, a_{22}, a_{33}, a_{44} \rightarrow$ which are just the diagonal elements

{Note: recall the property of Eigen value, "The Eigen value of triangular matrix are just the diagonal elements of the matrix"}

Example: The Eigen vectors of $\begin{bmatrix} 1 & 3 \\ 2 & 0 \end{bmatrix}$ is -------------- (Here C is scalar)

(A)C(2, 3) (B) C (-2, 3) (C) C (2, $\frac{2}{3}$) (D) C (1, $\frac{2}{3}$)

Solution: $A = \begin{bmatrix} 1 & 3 \\ 2 & 0 \end{bmatrix}$, $|A - \lambda I| = \begin{vmatrix} 1-\lambda & 3 \\ 2 & -\lambda \end{vmatrix} = 0 \Rightarrow -\lambda(1-\lambda) - 6 = 0$

$\lambda^2 - \lambda - 6 = 0$ $(\lambda - 3)\ (\lambda+2) = 0$ $\lambda = 3, -2$

For $\lambda = 3$, $\begin{bmatrix} -2 & 3 \\ 2 & -3 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = 0 \Rightarrow -2x_1 + 3x_2 = 0$, $\frac{x_1}{3} = \frac{x_2}{2} \Rightarrow$ Eigen vector: C(3, 2)

For $\lambda = -2$, $\begin{bmatrix} 3 & 3 \\ 2 & 2 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = 0$ $3x_1 + 3x_2 = 0$, $\frac{x_1}{-1} = \frac{x_2}{1} \Rightarrow$ Eigen vector : C(-1, 1)

Option (D) is correct

Example: An n × n homogenous system of equation A X = 0 is given, the rank r < n then the system has

a) n - r independent solution
b) r independent solution
c) No solution
d) n - 2r independent solution

Solution: a)

Example: Find Eigen value & Eigen vector of $\begin{bmatrix} 1 & 1 & 3 \\ 1 & 5 & 1 \\ 3 & 1 & 1 \end{bmatrix}$

Solution: $|A - l I| = 0$,

$$\begin{vmatrix} 1-l & 1 & 3 \\ 1 & 5-l & 1 \\ 3 & 1 & 1-l \end{vmatrix} = 0,$$

$\lambda^3 - 7\lambda^2 + 36 = 0$,

$(\lambda + 2)(\lambda^2 - 9l + 18 = 0$

$\lambda = -2, 3, 6$

<u>Calculation of Eigen vector :</u>

For $\lambda = -2 \rightarrow$ $[A - \lambda I] X = \begin{bmatrix} 3 & 1 & 3 \\ 1 & 7 & 1 \\ 3 & 1 & 3 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = 0$,

$3x + y + 3z = 0$ ---------------------------------------(i)

$x + 7y + z = 0$ ---------------------------------------(ii)

$\frac{x}{-20} = \frac{y}{0} = \frac{z}{20}$

$\frac{x}{-1} = \frac{y}{0} = \frac{z}{1}$

Eigen vector (-1, 0, 1)

For $\lambda = 3 \rightarrow (1, -1, 1)$

$\lambda = 6 \rightarrow (1, 2, 1)$

Example: $A = \begin{bmatrix} 1 & 2 & -3 \\ 0 & 3 & 2 \\ 0 & 0 & -2 \end{bmatrix}$

Find Eigen value of $3A^3 + 5A^2 - 6A + 2I$

Solution: $|A - l I| = 0$

$$\begin{vmatrix} 1-l & 2 & -3 \\ 0 & 3-l & 2 \\ 0 & 0 & -2-l \end{vmatrix} = 0$$

$(1-\lambda)(3-\lambda)(-2-\lambda) = 0 \implies \lambda = 1, \lambda = 3, \lambda = -2$

Eigen value of $A = 1, 3, -2$

Eigen value of $A^3 = 1, 27, -8$

Eigen value of $A^2 = 1, 9, 4$

Eigen value of $I = 1, 1, 1$

First Eigen value of $3A^3 + 5A^2 - 6A + 2I = 3 \times 1 + 5 \times 1 - 6 \times 1 + 2$

$= 4$

Second Eigen value of $3A^3 + 5A^2 - 6A + 2I = 3(27)+5(9) - 6(3)+2$

$= 81 + 45 - 18 + 2$

$= 110$

Third Eigen value of $3A^3 + 5A^2 - 6A + 2I = 3(-8) + 5(4) - 6(-2) + 2$

$= -24 + 20 + 12 + 2$

$= 10$

Example: Find Eigen values of

$$A = \begin{bmatrix} a_{11} & 0 & 0 & 0 \\ a_{21} & a_{22} & 0 & 0 \\ a_{31} & a_{32} & a_{33} & 0 \\ a_{41} & a_{42} & a_{42} & a_{44} \end{bmatrix}$$

Solution:

$$|A-\lambda I| = \begin{vmatrix} a_{11-l} & 0 & 0 & 0 \\ a_{21} & a_{22-l} & 0 & 0 \\ a_{31} & a_{32} & a_{33-l} & 0 \\ a_{41} & a_{42} & a_{42} & a_{44-l} \end{vmatrix} = 0$$

Expanding, $(a_{11-l})(a_{22-l})(a_{33-l})(a_{44-l}) = 0$

$\lambda = a_{11}, a_{22}, a_{33}, a_{44} \rightarrow$ which are just the diagonal elements

Example: Find A^{-1} using Cayley Hamilton Theorem

$$A = \begin{bmatrix} 1 & 2 & -2 \\ 1 & 1 & 1 \\ 1 & 3 & -1 \end{bmatrix}$$

Solution: $|A - \lambda I| = 0$, $\begin{vmatrix} 1-l & 2 & -2 \\ 1 & 1-l & 1 \\ 1 & 3 & -1-l \end{vmatrix} = 0$

Expanding and simplifying, $\lambda^3 - \lambda^2 - 4l + 4 = 0$

From Cayley Hamilton theorem, $A^3 - A^2 - 4A + 4I = 0$

$A^2 - A - 4I + 4A^{-1} = 0$

$4A^{-1} = [-A^2 + A + 4I]$

$$A^2 = \begin{bmatrix} 1 & -2 & -2 \\ 3 & 6 & -2 \\ 3 & 2 & 2 \end{bmatrix}$$

$$4A^{-1} = -\begin{bmatrix}1 & -2 & -2\\ 3 & 6 & -2\\ 3 & 2 & 2\end{bmatrix} + \begin{bmatrix}1 & 2 & -2\\ 1 & 1 & 1\\ 1 & 3 & -1\end{bmatrix} + 4\begin{bmatrix}1 & 0 & 0\\ 0 & 1 & 0\\ 0 & 0 & 1\end{bmatrix}$$

$$A^{-1} = \frac{1}{4}\begin{bmatrix}4 & 4 & -4\\ -2 & -1 & 3\\ -2 & 1 & 1\end{bmatrix}$$

Note: For finding Eigen value, you can't use elementary transforms.

Example: If $A = \begin{bmatrix}2 & 1 & 1\\ 0 & 1 & 0\\ 1 & 1 & 2\end{bmatrix}$, Find the value of

$A^8 - 5A^7 + 7A^6 - 3A^5 + A^4 - 5A^3 + 8A^2 - 2A + I$.

Solution: $\begin{vmatrix}2-l & 1 & 1\\ 0 & 1-l & 0\\ 1 & 3 & 2-l\end{vmatrix} = 0$

$$\lambda^3 - 5\lambda^2 + 7l - 3 = 0$$

$$A^3 - 5A^2 + 7A - 3I = 0$$

Now, $A^8 - 5A^7 + 7A^6 - 3A^5 + A^4 - 5A^3 + 8A^2 - 2A + I$

$= A^5(A^3 - 5A^2 + 7A - 3I) + A(A^3 - 5A^2 + 7A - 3I) + A^2 + A + I$

$= 0 + 0 + A^2 + A + I$

$$= \begin{bmatrix}5 & 4 & 4\\ 0 & 1 & 0\\ 4 & 4 & 5\end{bmatrix} + \begin{bmatrix}2 & 1 & 1\\ 0 & 1 & 0\\ 1 & 1 & 2\end{bmatrix} + \begin{bmatrix}1 & 0 & 0\\ 0 & 1 & 0\\ 0 & 0 & 1\end{bmatrix}$$

$$= \begin{bmatrix}8 & 5 & 5\\ 0 & 3 & 0\\ 5 & 5 & 8\end{bmatrix}$$

Example: $A = \begin{bmatrix}1 & 0\\ 2 & 4\end{bmatrix}$, The Eigen values of $4A^{-1} + 3A + 2I$ are

(A) 6, 15 (B) 9, 12 (C) 9, 15 (D) 7, 15

Solution: $|A - \lambda I| = 0$

$\begin{vmatrix}1-l & 0\\ 2 & 4-l\end{vmatrix} = 0$, $(1-l)(4-l) = 0$

$\lambda^2 - 5l + 4 = 0 \qquad \Rightarrow \lambda = 1, 4$

$A^2 - 5A + 4I = 0$

$A - 5I + 4A^{-1} = 0$

$4A^{-1} = -A + 5I$----------------------------------(i)

Now Using the result of (i) $4A^{-1} + 3A + 2I = -A + 5I + 3A + 2I$

$= 2A + 7I$

1st Eigen value, $= 2\times 1 + 7 = 9$

2nd Eigen value, $= 2\times 4 + 7 = 15$

Example: Find the Eigen vector of A = $\begin{bmatrix} -5 & 2 \\ 2 & -2 \end{bmatrix}$

Solution: Eigen values are λ = -1, -6 (Calculate yourself)

For λ_1= -1 $(A-\lambda I)\,X = 0 \Rightarrow -4x_1 + 2x_2 = 0 \Rightarrow x_2 = 2x_1$

$2x_1 - x_2 = 0 \Rightarrow x_2 = 2x_1$

Put $x_1 = 1$, $x_2 = 2$, $\Rightarrow$ $X_1 = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$

$A X_1 = \begin{bmatrix} -5 & 2 \\ 2 & -2 \end{bmatrix}\begin{bmatrix} 1 \\ 2 \end{bmatrix} = \begin{bmatrix} -1 \\ -2 \end{bmatrix} = (-1)\begin{bmatrix} 1 \\ 2 \end{bmatrix} = \lambda_1 X_1$

Eigen Value Eigen Vector

Similarly for 2nd Eigen value λ_2 = -6

$(A - \lambda I)\,X = 0$ $x_1 + 2x_2 = 0 \Rightarrow x_2 = x_1/2$

$2x_1 + 4x_2 = 0 \Rightarrow x_2 = x_1/2$

Put $x_1 = 2$, $x_2 = -1$ $\Rightarrow$ $X_2 = \begin{bmatrix} 2 \\ -1 \end{bmatrix}$

$A X_2 = \begin{bmatrix} -5 & 2 \\ 2 & -2 \end{bmatrix}\begin{bmatrix} 2 \\ -1 \end{bmatrix} = \begin{bmatrix} -12 \\ -6 \end{bmatrix} = (-6)\begin{bmatrix} 2 \\ -1 \end{bmatrix} = \lambda_2 X_2$

Example: Find the Eigen value of

$A = \begin{bmatrix} a & b \\ -b & a \end{bmatrix}$

Solution: Eigen value = $|A - \lambda I| = 0$ $= \begin{vmatrix} a-\lambda & b \\ -b & a-\lambda \end{vmatrix} = 0$

$(a-\lambda)^2 + b^2 = 0$,

$a - \lambda = \pm bi$

$\lambda = a \pm bi$

Example: If l_1, l_2 --- l_4 are eigen value of A, Find Eigen value of $(A-\lambda I)^2$

Solution: $(A-\lambda I)^2 = A^2 - 2l\,AI + \lambda^2 + I^2$

$= A^2 - 2lA + \lambda^2 I$

Eigen value of $A^2 \rightarrow \lambda_1^{\,2}, \lambda_2^{\,2}$ .... $\lambda_n^{\,2}$

Eigen value of $2\lambda A \rightarrow 2\lambda\,\lambda_1, 2\lambda\,\lambda_2$ - - - - - $2\lambda\lambda_n$

Eigen value of $\lambda^2 I \rightarrow \lambda^2, \lambda^2$ ....λ^2

Eigen value of $A^2 - 2\lambda A + \lambda^2 I = l_1^{\,2} - 2\lambda\,\lambda_1 + \lambda^2,\ l_2^{\,2} - 2\lambda\,\lambda_2 + \lambda^2$, ------------ $l_n^{\,2} - 2\lambda\,\lambda_n^{\,2} + \lambda^2$

$= (\lambda_1 - l)^2, (\lambda_2 - l)^2$ - - - - - - - $(\lambda_n - l)^2$

Note: The characteristic roots of triangular matrix are just the diagonal element of the matrix.